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MAXIMUM THEOREMS AND REFLECTIONS OF SIMPLE WAVES

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SUMMARY

The solutions corresponding to the reflection of a centered simple wave along a straight wall and along a free streamline of constant pressure are formulated in mathematical terms and expressed in terms of a Fourier transform. It is shown that these solutions are simply related to some important solutions arising in the theory of linear partial differential equations. Moreover, it is found that the classical properties of these flows are closely related to some important theorems predicting "a priori bounds" for special mathematical problems, in such a way that it can be said that these properties are a physical interpretation of those theorems. Finally, it is pointed out briefly that these results lead also to some interesting observations concerning the mathematical theory of positive definite functions.

INTRODUCTION

This paper is concerned with two simple and fundamental notions. The first is the "centered simple wave" or "Prandtl-Meyer corner flow" and its reflections along a straight wall or a free streamline of constant pressure. This is a problem of great importance in the two-dimensional supersonic-flow theory (ref. 1), and the following qualitative result is very classical: An expansion simple wave is reflected as an expansion wave by a straight wall but reflected as a compression wave by a free streamline of constant pressure. However, it does not seem that any attempt has been made so far to give a mathematical formulation to this problem. It will be shown that these "elementary interactions" are closely related to some important solutions of the linear partial differential equation which is satisfied by the stream function when the hodograph method is used.

The second notion considered in this paper is the so-called maximum theorem, or more precisely "a priori estimate," for some specific problems related to a partial differential equation of the hyperbolic type. More precisely, the paper of Bers (ref. 2) devoted to the Cauchy problem is referred to, in which very simple bounds for the solution have been given. Consideration is also given to the maximum theorem relative to another type of boundary-value problem, which has been formulated in a special case in reference 3 and under much more general conditions in reference 4 by Agmon, Niremberg, and Protter. It is possible to show that a physical interpretation can be attached to these results by considering the elementary interactions previously mentioned. Moreover, it will be seen that some of the statements can be somewhat improved. Finally, the relation between these results and the mathematical theory of positive definite functions will be pointed out, and this relation seems to lead to new results in this mathematical field.

This work was conducted at Brown University under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

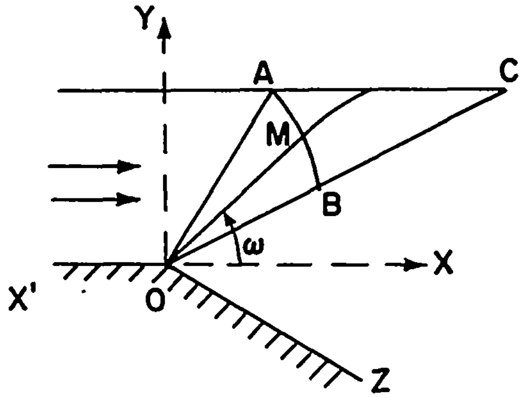
SYMBOLS

$D(\sigma,\sigma_1,\alpha)$	Fourier transform of $s(\theta,\sigma,\sigma_1)$	
$d(\theta,\sigma,\sigma_1)$	doublet at $\theta = 0$ and $\sigma = \sigma_1$	$(\sigma_1 \leqq 0)$
$H(\sigma,\sigma_1,\alpha)$	Fourier transform of $d(\theta,\sigma,\sigma_1)$	
$k(\sigma)$	defined by equation (7)	
M	Mach number	
m	mass flow, ρq	
q	magnitude of velocity vector	
r,ω	polar coordinates in physical plane	
$s(\theta,\sigma,\sigma_1)$	source at $\theta = 0$ and $\sigma = \sigma_1$	$(\sigma_1 \leqq 0)$
$t = \tan \mu$		
X,Y	Cartesian coordinates in physical plane	
α	dual variable of θ associated in Fourier transform	
θ	angle of velocity vector with X-axis	
μ	Mach angle	

ρ	density
σ	reduced speed, defined by equation (5)
φ	velocity potential
ψ	stream function

GENERAL EQUATIONS

Consider a uniform supersonic steady flow deflected along the corner X'OZ in sketch 1. As is known, a simple wave centered in O follows



Sketch 1

the uniform flow. The velocity vector is constant along any ray starting from O. In the physical plane the polar coordinates (origin O) r and ω will be introduced. Denote the magnitude of the velocity vector by q, its angle with OX by θ, the density by ρ, the mass flow by $M = \rho q$, the Mach number by M, the Mach angle by μ, and the stream function by ψ. Inside the simple wave ρ, q, m, M, and μ are functions of ω alone. The rays are characteristic lines and consequently make the angle μ with the streamlines. Then it is possible to write

$$\psi = Cmr \sin \mu \tag{1}$$

where C is a numerical constant, or

$$\frac{d\psi}{\psi} = \frac{dm}{m} + \frac{dr}{r} + \cot \mu \, d\mu \tag{2}$$

Now, according to the definition of the Mach angle μ, along a streamline

$$\frac{dr}{r} = -\cot \mu \; d\omega$$

and along a curvilinear characteristic

$$\frac{dr}{r} = -\cot 2\mu \; d\omega$$

As a result, by writing equation (2) along a streamline, it is found that

$$\frac{dm}{m} + \cot \mu \; d\mu = \cot \mu \; d\omega$$

and, along a curvilinear characteristic,

$$\frac{d\psi}{\psi} = (\cot \mu - \cot 2\mu) \; d\omega \tag{3}$$

Let $t = \tan \mu$ and $\cot 2\mu = \frac{1 - t^2}{2t}$ and equation (3) can be rewritten as

$$\frac{d\psi}{\psi} = \frac{1 + t^2}{2t} \, d\omega \tag{4}$$

This relation gives the value of ψ along a curvilinear characteristic of the simple wave. Now recall some classical material on the hodograph method. A more convenient variable σ is introduced instead of q, defined by

$$\sigma = \int_q^1 \frac{\rho}{q} \, dq \tag{5}$$

if the sonic speed is taken as the unit speed. Then it can be shown (ref. 5) that, when expressed in the hodograph variables σ and θ, the velocity potential φ and the stream function ψ of any flow satisfy the system.[1]

$$\left.\begin{aligned} \varphi_\theta &= -\psi_\sigma \\ \varphi_\sigma &= k(\sigma)\psi_\theta \end{aligned}\right\} \tag{6}$$

[1]Subscripts are used for partial derivatives.

where

$$k(\sigma) = \frac{1 - M^2}{\rho^2} \tag{7}$$

Then ψ is a solution of

$$k(\sigma)\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0 \tag{8}$$

Notice that $k(\sigma)$ has the sign of σ. To every solution of equation (8) there corresponds a flow in the physical plane, the correspondence being given by integration of the exact differential

$$dX + i\,dY = \exp\,(i\theta)\left(\frac{d\varphi}{q} + i\,\frac{d\psi}{m}\right) \tag{9}$$

Equation (4) may now be written in terms of this notation. First of all, for supersonic flow or for σ negative,

$$k(\sigma) = -\rho^{-2}t^{-2}$$

Now

$$\frac{dt}{d\sigma} = \left(1 + t^2\right)\frac{d\psi}{d\sigma} = -\frac{1}{\rho^2(-k)}\left[\rho_\sigma\left(\sqrt{-k}\right) + \rho\left(\sqrt{-k}\right)_\sigma\right] \tag{10}$$

The quantity ρ_σ may be computed from the Bernoulli equation

$$\rho M^2\,dq + q\,d\rho = 0$$

which shows, taking account of equations (5) and (7), that

$$\rho_\sigma = M^2 = 1 - k(\sigma)\rho^2 \tag{11}$$

Thus equation (10) becomes

$$\frac{1 + t^2}{2t}\frac{d\psi}{d\sigma} = -\frac{1 - k\rho^2}{2\rho} - \frac{\left(\sqrt{-k}\right)_\sigma}{2\sqrt{-k}} \tag{12}$$

The simple wave which has been considered is mapped on a line along which $d\theta = \sqrt{-k}\,d\sigma$. On the other hand, obviously $\omega = \theta + \mu$. Equation (4) gives

$$\frac{d\psi}{\psi\, d\sigma} = \frac{1 + t^2}{2t}\,\frac{d\theta + d\mu}{d\sigma} = \frac{1 + t^2}{2t}\sqrt{-k} -$$

$$\frac{1 - k\rho^2}{2\rho} - \frac{\left(\sqrt{-k}\right)_\sigma}{2\sqrt{-k}} = -\frac{1}{4}\,\frac{(-k)_\sigma}{(-k)}$$

The following theorem is thus obtained:

Theorem 1: The value of the stream function along a curvilinear characteristic of a centered simple wave is given by

$$\psi = C(-k)^{-1/4} \tag{13}$$

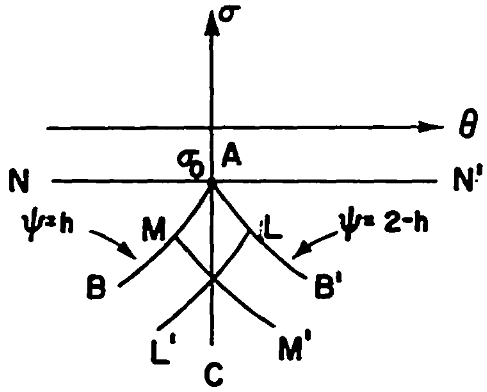
where C is a numerical constant.

REFLECTION OF A SIMPLE WAVE ALONG A WALL

Consider the reflection of the centered simple wave along a wall parallel to the X-axis (sketch 1). The velocity of the originally uniform supersonic flow is assumed to correspond to the value σ_o of the variable σ $\left(\sigma_o\right.$ is negative$\left.\right)$. This flow remains uniform until it reaches the characteristic OA. Denote by AMB the curvilinear characteristic of the simple wave which passes through A; it is desired to define ψ in the region ABC. Without any loss of generality, it can be assumed that $\psi = 1$ on the wall. In the hodograph plane $\theta\sigma$, ψ is a solution of equation (8) which is equal to 1 on AC and, according to equation (13), to $h(\sigma) = \left[\frac{k(\sigma_o)}{k(\sigma)}\right]^{-1/4}$ on AMB. Applying a symmetry argument in the $\theta\sigma$-plane with respect to the σ-axis, which corresponds to a symmetry condition with respect to the wall in the physical plane, a solution of equation (8) must be found with the two boundary conditions along the characteristics AB and AB' (sketch 2),

$$\psi = h(\sigma) \quad \text{along AB}$$

$$\psi = 2 - h(\sigma) \quad \text{along AB'}$$



Sketch 2

This problem will be solved by using a Fourier transform and it will be proved at the same time that this solution is closely related to the Riemann function of equation (8).

Define the Fourier transform of ψ by the pair of formulas

$$\left.\begin{aligned} U(\alpha,\sigma) &= \int_{-\infty}^{\infty} \psi(\theta,\sigma) \exp(-2i\pi\alpha\theta)\, d\theta = F(\psi) \\ \psi(\theta,\sigma) &= \int_{-\infty}^{\infty} U(\alpha,\sigma) \exp(2i\pi\alpha\theta)\, d\alpha = F^{-1}(U) \end{aligned}\right\} \tag{14}$$

Equation (8) is transformed into the ordinary differential equation

$$U_{\sigma\sigma} - 4\pi^2\alpha^2 k(\sigma)U = 0 \tag{15}$$

Now consider the solution $U = H(\sigma,\sigma_o,\alpha)$ of equation (15) which fulfills the boundary conditions

$$\left.\begin{aligned} H(\sigma_o,\sigma_o,\alpha) &= 1 \\ H_\sigma(\sigma_o,\sigma_o,\alpha) &= 0 \end{aligned}\right\} \qquad (16)$$

(it is an entire function in α according to Poincaré's theorem) and the solution ψ of equation (8) defined by

$$\psi = F^{-1}\left[(i\pi\alpha)^{-1}H(\sigma,\sigma_o,\alpha)\right] \qquad \sigma \leqq \sigma_o \qquad (17)$$

This solution is obtained by assuming that the integral is to be taken along a path which follows the real α-axis but passes below the origin in order to avoid the pole $\alpha = 0$. It has the following properties:

(1) For $\sigma = \sigma_o$, ψ is 0 for θ negative, $\psi = 2$ for θ positive, and ψ_σ is equal to 0. Consequently, ψ is identically 0 in the region NAB and identically equal to 2 in the region N'AB'.

(2) In order to investigate the properties of ψ inside the region BAB', the asymptotic behavior of H as a function of α for large values of $|\alpha|$ must be investigated. The following lemma will be proved later:

Lemma 1: For large values of $|\alpha|$,

$$H(\sigma,\sigma_o,\alpha) \approx h(\sigma) \cos 2\pi\alpha \left(y - y_o\right)$$

where y is related to σ by

$$dy = -\sqrt{-k(\sigma)}\, d\sigma \qquad (y = 0 \quad \text{for} \quad \sigma = 0)$$

(the same relation holds between y_o and σ_o).

Assuming the validity of this lemma, it is clear that ψ as defined by equation (17) is a continuously differential function of θ and σ inside the region BAB' which has some discontinuities when crossing the characteristics AB and AB'. When θ is increasing, ψ admits a jump equal to $h(\sigma)$ along AB $\left(\theta + y - y_o = 0\right)$ and along AB' $(\theta - y + y_o = 0)$. Consequently, the function ψ given by equation (17) is the solution of the problem because it is the unique continuous solution defined in BAB' which satisfies the boundary conditions along AB and AB'.

Now the relation between this solution and the Riemann function remains to be shown. It was shown (ref. 6) that the Riemann function for A and $\sigma < \sigma_0$ can be defined as $-F^{-1}\left[D\left(\sigma,\sigma_0,\alpha\right)\right]$ where $D\left(\sigma,\sigma_0,\alpha\right)$ is a solution of equation (15) which satisfies the conditions $D\left(\sigma_0,\sigma_0,\alpha\right) = 0$ and $D_\sigma\left(\sigma_0,\sigma_0,\alpha\right) = 1$. It was also shown that if $T(\sigma,\alpha)$ and $S(\sigma,\alpha)$ are the solutions of equation (15) which satisfy

$$S(0,\alpha) = 0$$

$$T(0,\alpha) = 1$$

$$S_\sigma(0,\alpha) = 1$$

$$T_\sigma(0,\alpha) = 0$$

it is possible to write

$$D\left(\sigma,\sigma_0,\alpha\right) = S\left(\sigma,\alpha\right)T\left(\sigma_0,\alpha\right) - S\left(\sigma_0,\alpha\right)T(\sigma,\alpha)$$

Similarly, one can write

$$H\left(\sigma,\sigma_0,\alpha\right) = T(\sigma,\alpha)S_\sigma\left(\sigma_0,\alpha\right) - S(\sigma,\alpha)T_\sigma\left(\sigma_0,\alpha\right)$$

because the right-hand side is a solution of equation (15) which satisfies conditions (16). Then

$$H\left(\sigma,\sigma_0,\alpha\right) = -D_{\sigma_0}\left(\sigma,\sigma_0,\alpha\right) \tag{18}$$

Taking account of equation (17), this result can be interpreted as follows:

Thorem 2: If $\psi\left(\theta,\sigma,\sigma_0\right)$ is the solution which corresponds to the reflection of a centered simple wave along a wall and $R\left(\theta,\sigma,0,\sigma_0\right)$ is the Riemann function of the equation (as previously defined), the following identity holds:

$$\psi_\theta\left(\theta,\sigma,\sigma_0\right) = 2R_{\sigma_0}\left(\theta,\sigma,0,\sigma_0\right) \tag{19}$$

Proof of lemma 1: The result of lemma 1 is a very classical one. For brevity, in order to avoid a new proof, it is suggested that equation (18) and the result given in reference 6 be used in order to derive the required property.

In the limiting case when σ_o tends towards 0, the given uniform flow becomes sonic. The function ψ is defined inside the region bounded by the characteristics OT and OT' passing through the origin of the $\theta\sigma$-plane. It is equal to 0 on OT and to 2 on OT', and its derivative with respect to θ is twice the solution previously called (refs. 3 and 7) the minus-doublet. In this case no discontinuities appear along the characteristics OT and OT'.

INTERPRETATION AND IMPROVEMENT OF BERS' THEOREM

The Bers theorem considered herein is concerned with the Cauchy problem with data on a segment PQ of the line $\sigma = \sigma_1$, σ_1 being a nonpositive constant. On PQ it is assumed that

$$\left.\begin{aligned} \psi(\theta,\sigma_1) &= \tau(\theta) \\ \psi_\sigma(\theta,\sigma_1) &= \nu(\theta) \end{aligned}\right\} \tag{20}$$

$\tau(\theta)$ and $\nu(\theta)$ being known functions. These data define the solution in a domain PQR, lying in the region $\sigma < \sigma_1$ and bounded by the two concurrent characteristics PR and QR. Bers' theorem may be stated as follows: Provided that $k(\sigma)$ is an increasing, piecewise continuous function of σ, the solution ψ of the Cauchy problem satisfies in PQR the following inequality

$$|\psi| < M + N(\sigma_1 - \sigma) \tag{21}$$

where M and N are, respectively, the bounds of $|\tau(\theta)|$ and $|\nu(\theta)|$ on the segment PQ.

This theorem may be broken down into two parts: (1) When the special case $\nu(\theta) = 0$ is considered and (2) when $\tau(\theta) = 0$. It is obvious that, according to the linearity of the solution with respect to the data, relation (21) is implied by the results corresponding to the two parts.

Define $d(\theta,\sigma,\sigma_1)$ and $s(\theta,\sigma,\sigma_1)$ as the doublet and the source at $\theta = 0$ and $\sigma = \sigma_1$, that is, the solutions of equation (8) defined for $\sigma < \sigma_1$ and for every value of θ, so that $d_\sigma(\theta,\sigma_1,\sigma_1) = 0$ and $s(\theta,\sigma_1,\sigma_1) = 0$ and that the values of d and s_σ along $\sigma = \sigma_1$ reduce to the Dirac distributions[2] at $\theta = 0$. Equivalent definitions are

[2] d and s are 0 outside the two characteristics passing through $\theta = 0$ and $\sigma = \sigma_1$.

$$\left.\begin{aligned} d(\theta,\sigma,\sigma_1) &= F^{-1}\left[H(\sigma,\sigma_1,\alpha)\right] \\ s(\theta,\sigma,\sigma_1) &= F^{-1}\left[D(\sigma,\sigma_1,\alpha)\right] \end{aligned}\right\} \tag{22}$$

It will be shown that Bers' theorem is implied by the following theorem:

Theorem 3: If $k(\sigma)$ is an increasing, piecewise continuous function, then d is nonnegative and s, nonpositive.

Assume that this result is true; then an improvement of Bers' theorem can be derived immediately. Consider the first part, $\nu(\theta) = 0$. The solution is obtained as a superposition of doublets. If d is nonnegative, $\tau(\theta) > 0$ implies that $\psi > 0$. As for $\tau = 1$ and $\psi = 1$, it can be concluded that:

If along PQ $\nu(\theta) = 0$ and $M' \leqq \tau(\theta) \leqq M$, then in PQR $M' < \psi < M$. This is a maximum theorem which gives a slightly more precise result than the first part of Bers' theorem.

Similarly, if $\tau = 0$, the solution is obtained as a superposition of the sources. If $\nu(\theta) > 0$, then $\psi < 0$. On the other hand, if $\nu = 1$, $\psi = \sigma - \sigma_1$ and, consequently:

If along PQ $\tau(\theta) = 0$ and $N' \leqq \nu(\theta) \leqq N$, then in PQR $N(\sigma - \sigma_1) \leqq \psi \leqq N'(\sigma - \sigma_1)$. To simplify the proof of theorem 3 consider the partial statement of theorem 3':

Theorem 3': For the class of functions $k(\sigma)$ considered in theorem 3, d is not negative.

It can be proved that theorem 3 is a direct consequence of theorem 3'. Namely, for any $\sigma_0 \leqq \sigma_1$ and any $\sigma \leqq \sigma_0$, $F^{-1}\left[H(\sigma,\sigma_0,\alpha)\right] \leqq 0$. Now, according to equation (18),

$$\int_{\sigma}^{\sigma_1} H(\sigma,\sigma_0,\alpha)\, d\sigma_0 = -D(\sigma,\sigma_1,\alpha)$$

Thus $F^{-1}\left[D(\sigma,\sigma_1,\alpha)\right] = s(\theta,\sigma,\sigma_1) \leqq 0$ according to the linearity of the Fourier transform; theorem 3 is therefore a consequence of its first part (theorem 3').

Now, it will be shown that theorem 3' - and consequently theorem 3 and Bers' theorem - can be interpreted as a direct mathematical consequence of the following very well known result concerning the reflection of a simple wave along a straight wall:

An expansion simple wave is reflected as an expansion wave; in particular, no limiting line can occur in the region where the reflection takes place.

In other words, any mathematical proof of the previous statement implies, as a particular consequence, the Bers theorem.

In order to prove this implication note that, along a characteristic $d\theta = \epsilon \sqrt{-k}\, d\sigma$ $(\epsilon = \pm 1)$, one has, according to equation (6),

$$
\begin{aligned}
d\varphi &= \left[k(\sigma)\psi_\theta - \epsilon\psi_\sigma \sqrt{-k}\right] d\sigma \\
&= -\epsilon \sqrt{-k}\, d\psi
\end{aligned}
$$

and, according to equations (9) and (7),

$$
\begin{aligned}
dX + i\, dY &= \exp\ (i\theta)\ d\psi \left(-\epsilon \frac{\sqrt{-k}}{q} + \frac{i}{m}\right) \\
&= \exp\ (i\theta)\ d\psi \left(- \frac{\epsilon}{\rho q} \cot \alpha + \frac{i}{m}\right) \\
&= \exp\ \left[i(\theta - \epsilon\alpha)\right] \frac{d\psi}{m \sin \alpha}
\end{aligned}
\tag{23}
$$

As a result, ψ is a monotonic function along each characteristic. More precisely, along the characteristics MM' and L'L (sketch 2), ψ is an increasing function of θ. Then ψ_θ is positive in BAB'; but

$$
\psi_\theta\left(\theta,\sigma,\sigma_0\right) = 2R_{\sigma_0}\left(\theta,\sigma,0,\sigma_0\right) = 2d(\theta,\sigma,\sigma_0)
$$

and this result is equivalent to theorem 3'.

Two proofs for this theorem will be outlined briefly. Proof (a) is a complement of the Bers proof (assuming the Bers theorem to be known); proof (b) is a direct proof.

Proof (a): Assume the following Cauchy data: $\nu_1(\theta) = 0$ and $\tau_1(\theta) = 1$ on a segment PQ of the line $\sigma = \sigma_1$ and 0 outside.[3] According to Bers' theorem, for every $\sigma < \sigma_1$ the corresponding solution ψ_1 satisfies $-1 < \psi_1 < 1$.

Now consider ψ_2 corresponding to the data $\nu_2(\theta) = 0$ and $\tau_2(\theta) = 1 - \tau_1$; the similar result $-1 < \psi_2 < 1$ is obtained; but $\psi_1 + \psi_2 = 1$. Thus $0 < \psi_1 < 1$. Then consider the case $\nu_o(\theta) = 0$ and $\tau_o(\theta) = 0$ for $\theta < 0$ and $\tau_o(\theta) = 1$ for $\theta > 1$. It must be proved that for every $\sigma < \sigma_1$ the corresponding $\psi_o(\theta,\sigma)$ is a non-decreasing function of θ. Assume for a moment that the result is false; then for $\sigma = \sigma_2$ there exist two numbers θ_1 and θ_2 $\left(\theta_1 < \theta_2\right)$ such that

$$\psi_o\left(\theta_1,\sigma_2\right) > \psi_o\left(\theta_2,\sigma_2\right)$$

According to the above relation, $\psi_o(\theta,\sigma) - \psi_o\left(\theta + \theta_2 - \theta_1,\sigma\right)$ is a solution of equation (8) which is negative for $\sigma = \sigma_2$ and $\theta = \theta_1$; but this function is a particular function ψ_1 considered in the beginning of the proof, and it was shown that ψ_1 is positive. Thus theorem 3' is proved.

Proof (b): Following the method used by Bers, consider first the case when $k(\sigma)$ is a step function:

$$k(\sigma) = -\omega_n^2 \quad \text{for} \quad \sigma_{n+1} < \sigma < \sigma_n \qquad (n = 1,\ 2,\ .\ .\ .)$$

ω_n being a sequence of increasing numbers. The following lemma will be proved:

Lemma 2: When $k(\sigma)$ is such a step function, $d\left(\theta,\sigma,\sigma_1\right)$ is a positive distribution.

Consider, in this case, $H\left(\sigma,\sigma_1,\alpha\right)$. For $\sigma_2 \leqq \sigma \leqq \sigma_1$, $H = \cos\left(2\pi\alpha\omega_1\sigma\right)$. Now assume that, for $\sigma_{n+1} \leqq \sigma \leqq \sigma_n$,

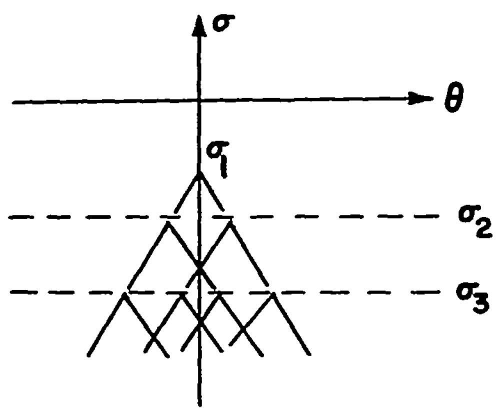
[3]It can be shown that no special difficulties occur on account of the discontinuity in $\tau(\theta)$.

$H = \sum_p \alpha_{p,n} \exp\left[2i\pi\alpha\omega_n(\sigma - s_{p,n})\right] + \beta_{p,n} \exp\left[-2i\pi\alpha\omega_n(\sigma - s_{p,n})\right]$, $\alpha_{p,n}$, $\beta_{p,n}$, and $s_{p,n}$ being suitable real constants, $\alpha_{p,n}$ and $\beta_{p,n}$ being positive, and $\sum_p$ denoting a finite sum. It is easy to show that a similar result holds for $\sigma_{n+1} \leqq \sigma \leqq \sigma_{n+2}$, because the function $\exp\left[\epsilon 2i\pi\alpha\omega_n(\sigma - \sigma_{n+1})\right]$ (where $\epsilon = \pm 1$), a solution of equation (15) in the strip $\sigma_n \leqq \sigma \leqq \sigma_{n+1}$, must be continued in the strip $\sigma_{n+1} \leqq \sigma \leqq \sigma_{n+2}$ by

$$\frac{1}{2}\left(1 + \frac{\omega_n}{\omega_{n+1}}\right) \exp\left[\epsilon 2i\pi\alpha\omega_{n+1}(\sigma - \sigma_{n+1})\right] +$$

$$\frac{1}{2}\left(1 - \frac{\omega_n}{\omega_{n+1}}\right) \exp\left[-\epsilon 2i\pi\alpha\omega_{n+1}(\sigma - \sigma_{n+1})\right]$$

in order to have in $\sigma_n \leqq \sigma \leqq \sigma_{n+2}$ a continuously differentiable function with respect to σ. The presumed expansion for H is then valid because ω_n is increasing with n. It is easy to formulate the corresponding result in the $\theta\sigma$-plane. For every $\sigma < \sigma_1$, $d(\theta,\sigma,\sigma_1)$ is a distribution[4] in θ which is the sum of a finite number of Dirac distributions with positive coefficients. The important property that the coefficients $\alpha_{p,n}$ and $\beta_{p,n}$ are positive is used here. In other words, in the $\theta\sigma$-plane, for $\sigma < \sigma_1$, d is a positive measure equal to 0 everywhere except on the singular characteristics shown in sketch 3.



Sketch 3

[4]For the properties of distributions used in this paper, see reference 8.

Now theorem 3' is an immediate consequence of the following lemma:

Lemma 3: If a sequence of increasing step functions $k_n(\sigma)$ tends uniformly toward an increasing, piecewise continuous function $k(\sigma)$ in $\sigma_2 \leqq \sigma \leqq \sigma_1$, then the distribution doublets $d_n(\theta,\sigma,\sigma_1)$ tend toward $d(\theta,\sigma,\sigma_1)$ (in the sense of the theory of distributions).

As a result $d(\theta,\sigma,\sigma_1)$ is a positive distribution and, in particular, a positive function when d is a function.[5] This is precisely the statement of theorem 3'.

This lemma states the continuity of the solution of the Cauchy problem with respect to the coefficients of equation (8) - in fact, here, the function $k(\sigma)$. The result is a classical one when the data and, consequently, the corresponding solutions are "regular" enough (one possible proof is given in ref. 2 as an application of the Σ-monogenic-functions theory). Consider, for instance, the data $\tau(\theta) = 0$ for $\theta < 0$ and $\tau(\theta) = \theta^p/p!$ for $\theta > 0$ and $\nu(\theta) = 0$. The sequence of solutions for this problem corresponding to $k_n(\sigma)$ tends uniformly in every closed subdomain to the solution corresponding to $k(\sigma)$; but $d_n(\theta,\sigma,\sigma_1)$ and $d(\theta,\sigma,\sigma_1)$ are, respectively, the derivatives of order $p + 1$ of this sequence and its limit. The result of lemma 3 follows according to the continuity of the differentiation in distribution theory and, at the same time, the proof of theorem 3' is achieved.

It was noticed above that theorem 3' and Bers' theorem are consequences of the fundamental property of the reflection of a simple wave along a straight wall. However, it does not seem obvious that the converse is true. It should be noted, however, that the method used in proof (b) not only leads to the conclusion of theorem 3' but also gives a mathematical proof of the statement concerning this property of the reflection of a centered simple wave. The reader will recognize this fact easily after a brief inspection of the results of lemmas 2 and 3.

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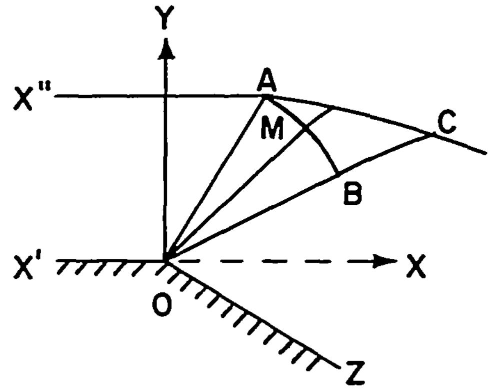
[5]In particular, when $k(\sigma)$ is continuous in $\sigma_2 \leqq \sigma \leqq \sigma_1$.

REFLECTION OF A SIMPLE WAVE ALONG A FREE STREAMLINE

Consider a uniform supersonic flow bounded by a straight wall X'O and a parallel free streamline X"A (sketch 4). This flow turns around



Sketch 4

the corner X'OZ. A centered simple wave follows the uniform flow. As a result, the free streamline is bent down and the initial expansion wave is reflected as a compression wave. It is desired to give a mathematical solution of this problem and to find the function ψ in the region of reflection ABC as function of σ and θ.

This region is mapped into the ABC triangle of the corresponding hodograph plane $\theta\sigma$; AB and AC are two characteristics. The following conditions must be satisfied: Along AC,

$$\psi = 1$$

and, along AB,

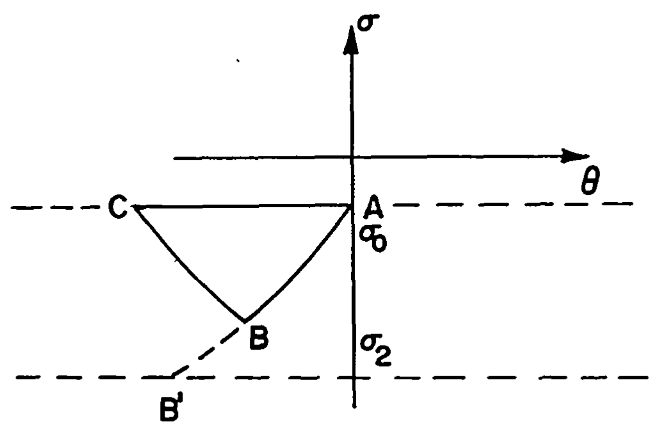
$$\psi = h(\sigma) = \left[\frac{k(\sigma_o)}{k(\sigma)}\right]^{1/4}$$

The last value was obtained in the section "General Equations"; σ_o is the value of σ corresponding to the given uniform incoming flow. Notice the special case $\sigma_o = 0$; in this case $\psi = 1$ along the sonic line $\sigma = 0$ and $\psi = 0$ along the characteristics. The importance of such a solution was pointed out previously (refs. 3 and 5).

An expression for the solution of this problem will be given first. If

$$\psi^* = F^{-1}\left[\frac{D(\sigma,\sigma_2,\alpha)}{2i\pi\alpha D(\sigma_o,\sigma_2,\alpha)}\right] \qquad (\sigma_2 < \sigma_o) \qquad (24)$$

the integration of equations (14) being carried out along a line $\operatorname{Im}(\alpha) = c$ where c is a small positive constant, then ψ^* is a solution of equation (8); when $\sigma = \sigma_o$, $\psi^* = 0$ for $\theta > 0$ and $\psi^* = 1$ for $\theta < 0$; when $\sigma = \sigma_2$, $\psi^* = 0$ for every value of θ. Moreover, it was shown (refs. 6 and 7) that ψ^* is identically 0 for any point of the strip $\sigma_2 \leqq \sigma \leqq \sigma_o$ lying on the right side of the characteristic ABB' (sketch 5) and the asymptotic behavior of the integrand in the Fourier



Sketch 5

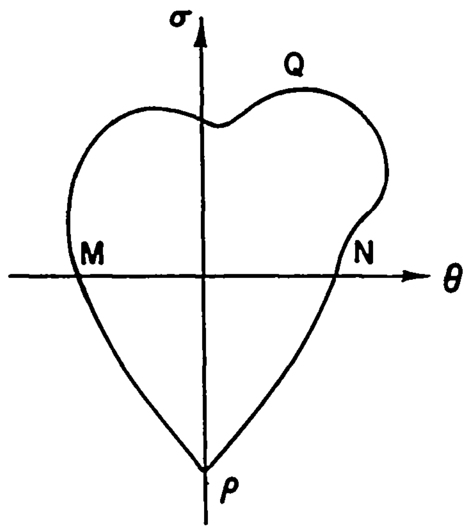
integral shows immediately that ψ^* admits a jump equal to $-h(\sigma)$ along this characteristic ABB'. Thus, provided that σ_2 is lower than the value σ_B of σ in B, ψ^* is identically equal to ψ in the region ABC where the solution is to be found. A result very similar to formula (19) of theorem 2 could be formulated, relating this solution ψ to Green's function of a strip as defined in reference 6. The introduction of the arbitrary quantity σ_2 is, in some sense, quite artificial. Such an introduction has been found necessary when $k(\sigma)$ is defined only for $\sigma > -\sigma_m$, σ_m being a positive quantity, as is the case for the usual ideal gas with constant specific heats. When the equation of state is such that $k(\sigma)$ is an increasing function defined for every negative value of σ, it is possible to consider, instead of equation (24), the solution

$$\psi = F^{-1}\left[(2i\pi\alpha)^{-1} H_1(\sigma, \sigma_0, \alpha)\right] \tag{25}$$

where H_1 is the solution of equation (15) equal to 1 for $\sigma = \sigma_0$ which, for $\alpha = i\alpha'$ and α' positive, is real and tends toward 0 when σ tends toward $-\infty$. (For a discussion of the existence and properties of such a function H_1, see ref. 9.) In fact, in every finite domain ψ is the limit of ψ^* when $\sigma_2 \rightarrow -\infty$.

Now the relation between the qualitative properties of such a flow and the maximum theorem of a Tricomi problem remains to be emphasized. For brevity, only the case $\sigma_0 = 0$ is considered in the following discussion. Assume first that the equation of state of the fluid and the turning angle X'OZ are such that no limiting line appears in the region ABC where the reflection takes place. According to equation (23), $k(\sigma)$ is such that the following property A is satisfied for some a such that $a < \sigma(B)$.

Property A: The solution of equation (8) equals 1 along the segment MN (sketch 6) of the θ-axis and 0 along the arc of the characteristic MP and varies monotonically along any arc of a characteristic drawn inside MNP, provided $a \leqq \sigma(P) < 0$, a being a negative constant.



Sketch 6

Now it was shown[6] in reference 3 that from property A a maximum theorem for the Tricomi problem can be derived: If the values of a solution ψ of equation (8), defined in the domain MQNPM, are equal to 0 on MP and are bounded in modulus by K along the arc MQN drawn in the half plane $\sigma > 0$ (sketch 6), then $|\psi| < K$ inside the domain MQNPM, provided $a \leqq \sigma(P)$.

Thus, a relation is established between the velocity for which a limiting line appears in the flow and the range of validity of the maximum theorem. Recently, Agmon, Niremberg, and Protter (ref. 4) have found sufficient conditions for the validity of property A. They proved that, if $k(\sigma)$ is an increasing function,[7] property A is valid with $a = a_o$, a_o being the largest negative root of the equation $e(\sigma) = \left\{\left[-k(\sigma)\right]^{1/4}\right\}_{\sigma\sigma} = 0$, and a_o must be taken as $-\infty$ if $e(\sigma) > 0$. Thus, this result gives the following information for the flow considered in sketch 4: The velocity for which a limiting line may occur is greater than the value of the velocity corresponding to $\sigma = a_o$. In other words, if a_m is the value of σ which corresponds to the first occurrence of a limiting line in the flow $\left(\text{when } \sigma_o = 0\right)$, then $a_m \leqq a_o$. The question whether a_m is effectively equal to a_o or less than a_o is still open. If the equality holds, then this gives a very simple way to compute the velocity at which the limiting line occurs. If not, a in property A can be chosen equal to a_m, which means that the range of validity of the maximum theorem as derived from reference 4 may be extended.

CORRESPONDING RESULTS IN THEORY OF POSITIVE DEFINITE FUNCTIONS

The relations between the results obtained in the preceding sections and mathematical theory of positive definite functions will now be pointed out. For the properties of such functions and for a bibliography of this topic, see reference 10.

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[6]The proof was given for the Tricomi equation but can be extended immediately to the case which is considered here.

[7]To be specific, it is assumed here that $k(\sigma) = \sigma c(\sigma)$ with $c(\sigma)$ a continuous positive function, twice continuously differentiable. Thus, $e(\sigma)$ is a positive function for sufficiently small values of σ.

A continuous function $f(x)$ defined for every real value of x is positive definite if, whatever be the real numbers $x_1 \ldots x_l$ and the complex numbers $z_1 \ldots z_l$, the following inequality holds:

$$\sum_{j,k} f(x_j - x_k) z_j \bar{z}_k \geqq 0$$

This definition was generalized by Schwartz (ref. 8), who has given the definition of a positive definite distribution. In this theory the fundamental theorem is Bochner's theorem which, as generalized by Schwartz,[9] states that a distribution is positive definite if and only if its Fourier transform is a nonnegative measure.

Now it is clear that theorem 3 gives rise immediately to a theorem which can be formulated as follows:

Theorem 4: Consider the differential equation

$$u_{zz} + \beta^2 p(z)u = 0 \tag{26}$$

where $p(z)$ is a nondecreasing function of z in the interval $z_1 \leqq z \leqq z_2$. Choose a such that $z_1 \leqq a \leqq z_2$ and consider the solutions $D_a(z,\beta)$ and $S_a(z,\beta)$ of equation (26) which satisfy the conditions

$$D_a(a,\beta) = 1$$

$$\frac{\partial}{\partial z} D_a(a,\beta) = 0$$

$$S(a,\alpha) = 0$$

$$\frac{\partial}{\partial z} S(a,\alpha) = 1$$

Then, for every value of z such that $a \leqq z \leqq z_2$, D_a and S_a, as functions of the real variable β, are positive definite.

Similarly, using the results of the section "Reflection of a Simple Wave Along a Free Streamline," the following theorem can be derived:

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[9]Schwartz's extension of the Fourier transform is used here.

Theorem 5: If $p(z)$ is a positive, nondecreasing function such that $\left[p(z)^{-1/4}\right]_{zz} \geqq 0$ for $z \geqq z_1$, then the solution $H_a(z,\beta)$ of equation (26) which is equal to 1 for $z = a$ and which tends toward 0 when β has an argument equal to $\pi/2$ and z tends toward infinity is, as a function of the real variable β, a positive definite function for any fixed value of z greater than a.

Applications of these two theorems can be written immediately for the special examples of equation (26) considered in reference 9.

Brown University,
Providence, R. I., September 20, 1954.

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